

THE GROTHENDIECK–TEICHMÜLLER GROUP AND THE STABLE SYMPLECTIC CATEGORY

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ABSTRACT. We consider an oriented version of the stable symplectic category defined in [27]. We define a canonical representation (or fiber functor) on this category, and study its motivic group of monoidal automorphisms. In particular, we observe that this Galois group contains a natural subgroup isomorphic to the abelian quotient of the Grothendieck–Teichmüller group. We also study the rational Waldhausen K-theory of the E_∞ -ring spectrum of coefficients Ω of the stable symplectic category, and its relation to the symplectomorphism group of an object in the stable symplectic category.

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1. INTRODUCTION

In [27] the first author defined a stabilization $h\mathbb{S}$ of the symplectic category introduced by A. Weinstein in [36, 38]. The objects of Weinstein’s category are symplectic manifolds, and the morphisms between two symplectic manifolds (M, ω) and (N, η) are lagrangian immersions into $\overline{M} \times N$, where the conjugate symplectic manifold \overline{M} is defined by the pair $(M, -\omega)$. The composition $L_1 * L_2$ of two lagrangian immersions $L_1 \looparrowright \overline{M} \times N$ and $L_2 \looparrowright \overline{N} \times K$, is defined to be the fiber product: $L_1 \times_N L_2 \longrightarrow \overline{M} \times K$. This definition does not always yield a lagrangian immersion to $\overline{M} \times K$: to do so, the pullback defining it must be transverse, so Weinstein’s construction is unfortunately not a genuine category.

In [27], we described a way to extend the symplectic category to an honest category $h\mathbb{S}$, by introducing a moduli space of stabilized (in the sense of homotopy theory) lagrangian immersions in a symplectic manifold of the form $\overline{M} \times N$ ¹. This moduli space can be described as the infinite loop space corresponding to a certain Thom spectrum. Taking this

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¹under the assumption of monotonicity. Otherwise, one has the space of totally real immersions.

as the space of morphisms defines a *stable symplectic (homotopy) category* $h\mathbb{S}$ that is naturally enriched over the homotopy category of spectra (under smash product)². Composition in $h\mathbb{S}$ is well-defined and remains faithful to Weinstein’s original definition. Geometrically, the stabilization of Weinstein’s category can be seen as “inverting the symplectic manifold \mathbb{C}^n ”, analogous to the introduction of an inverse to the projective line in the theory of motives. In other words, we introduce a relation on the symplectic category that identifies two symplectic manifolds M and N if $M \times \mathbb{C}^k$ becomes equivalent to $N \times \mathbb{C}^k$ for some k . This stable symplectic category has variants defined by lagrangian immersions with oriented and metaplectic structures.

Here we study a monoidal functor from a closely related stable **oriented** symplectic category $s\mathbb{S}$ to the category of modules over a naturally associated ring spectrum $s\Omega$ described as a Thom spectrum $(U/SO)^{-\zeta}$. On a symplectic manifold M the value of this functor is an $s\Omega$ -module $s\Omega(M)$ representing the space of stably immersed oriented lagrangians in M . By extending coefficients to other algebras over $s\Omega$, one has a family of such functors, and we can ask for the structure of the Galois group of monoidal automorphisms of this family. We answer this question (see corollary 4.6 and theorem 4.9), and relate this motivic group to the Grothendieck–Teichmüller group [30]. We also study the rational Waldhausen K-theory of $s\Omega$; in particular, we construct an interesting invariant of the classifying space of the symplectomorphism group of an object (M, ω) with values in the Waldhausen K-theory of $s\Omega$.

This document is organized as follows: In section 2 we recall the construction of the stable symplectic homotopy category. Section 3 describes a computation of the rational Waldhausen K-theory of the ring spectrum $s\Omega$, with applications to the symplectomorphism group. In section 4 we describe the algebraic representations that we are interested in, and identify the Galois group of monoidal automorphisms of its rationalization. Section 4 also describes the construction of an integral model for this Galois group. Section 5 summarizes some properties of the Grothendieck–Teichmüller group and related constructions, and applies ideas from Koszul duality (over \mathbb{Q}). In particular, we define a Hopf–Galois analog of the Grothendieck–Teichmüller group in the category of spectra, and identify its abelianization, over \mathbb{Q} , with the group of symmetries identified in Section 4.

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2. THE STABLE SYMPLECTIC HOMOTOPY CATEGORY

In this section we recall the construction of the stable symplectic homotopy category [27]. Given a symplectic manifold (M, ω) of real dimension $2m$, we construct a spectrum $\Omega(M)$ so that the corresponding infinite loop space can be interpreted as a space whose points represent manifolds that admit totally-real immersions into $M \times \mathbb{C}^n$ for large values of n (up to an equivalence that shall be made precise later). We will say that M satisfies monotonicity if the cohomology class of the symplectic form ω is a scalar multiple of the first Chern class of M . Under the assumption of monotonicity, totally real immersions may be replaced by lagrangian immersions in the above interpretation of $\Omega(M)$.

²In [27], we lifted this category to an A_∞ -category enriched over spectra.

§2.1 The basic construction

Consider the Thom spectrum $\Sigma^n \mathcal{G}(\tau \oplus \mathbb{C}^n)^{-\zeta_n}$, where the bundle ζ_n is defined by virtue of the pullback diagram:

$$\begin{array}{ccc} \mathcal{G}(\tau \oplus \mathbb{C}^n) & \xrightarrow{\zeta_n} & \mathrm{BO}(m+n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\tau \oplus \mathbb{C}^n} & \mathrm{BU}(m+n), \end{array}$$

where τ denotes (homotopy unique) complex structure on the tangent bundle of M compatible with the symplectic form ω . In [27], we used the work of D. Ayala [4] to show that for $n > 0$, the infinite loop space $\Omega^{\infty-n}(\mathcal{G}(\tau \oplus \mathbb{C}^n)^{-\zeta_n})$ can be interpreted as the moduli space of manifolds $L^{m+n} \subset \mathbb{R}^\infty \times \mathbb{R}^n$, with a proper projection onto \mathbb{R}^n , and endowed with a totally-real immersion $L^{m+k} \looparrowright M \times \mathbb{C}^n$ (or lagrangian immersion, under the assumption of monotonicity). More precisely, the space $\Omega^{\infty-n}(\mathcal{G}(\tau \oplus \mathbb{C}^n)^{-\zeta_n})$ is uniquely defined by the property that given a smooth manifold X , the set of homotopy classes of maps $[X, \Omega^{\infty-n}(\mathcal{G}(\tau \oplus \mathbb{C}^n)^{-\zeta_n})]$, is in bijection with concordance classes of smooth manifolds $E \subset X \times \mathbb{R}^\infty \times \mathbb{R}^n$ over X , so that the first factor projection: $\pi : E \rightarrow X$ is a submersion, and which are endowed with a smooth map $\varphi : E \rightarrow M \times \mathbb{C}^n$ which restricts to a totally-real immersion (resp. lagrangian) on each fiber of π . As before, we demand that the third factor projection $E \rightarrow \mathbb{R}^n$ be fiberwise proper over X .

Now the standard inclusion $\mathbb{R}^{n_1} \subseteq \mathbb{R}^{n_2}$, induces a natural map:

$$\varphi_{n_1, n_2} : \Sigma^{n_1} \mathcal{G}(\tau \oplus \mathbb{C}^{n_1})^{-\zeta_{n_1}} \rightarrow \Sigma^{n_2} \mathcal{G}(\tau \oplus \mathbb{C}^{n_2})^{-\zeta_{n_2}},$$

which represents the map that sends a concordance class E , to $E \times \mathbb{R}^{n_2-n_1}$, by simply taking the product with the orthogonal complement of \mathbb{R}^{n_1} in \mathbb{R}^{n_2} .

Definition 2.1. Define the Thom spectrum $\Omega(M)$ representing the infinite loop space of stabilized totally-real (resp. lagrangian under the assumption of monotonicity) immersions in M to be the colimit:

$$\Omega(M) = \underline{\mathcal{G}}(M)^{-\zeta} := \mathrm{colim}_n \Sigma^n \mathcal{G}(\tau \oplus \mathbb{C}^n)^{-\zeta_n}.$$

Notice that by definition, we have a canonical homotopy equivalence: $\Omega(M \times \mathbb{C}) \simeq \Sigma^{-1} \Omega(M)$. Taking M to be a point, we define $\Omega = \Omega(*) = (\mathrm{U}/\mathrm{O})^{-\zeta}$, where the bundle ζ over U/O is the virtual zero dimensional bundle over (U/O) defined by the canonical inclusion $\mathrm{U}/\mathrm{O} \rightarrow \mathrm{BO}$.

Henceforth, we shall use the term “lagrangian immersion” to mean “totally-real immersion” if the condition of monotonicity fails to hold. We take this opportunity to also introduce the (abusive) convention of not decorating the stable vector bundle ζ by the underlying manifold M . Hopefully, the manifold M will be clear from context.

We may also describe $\Omega(M)$ as a Thom spectrum: Let the stable tangent bundle of M of virtual (complex) dimension m be given by a map $\tau : M \rightarrow \mathbb{Z} \times \mathrm{BU}$. As the notation suggests, let $\underline{\mathcal{G}}(M)$ be defined as the pullback:

$$\begin{array}{ccc} \underline{\mathcal{G}}(M) & \xrightarrow{\zeta} & \mathbb{Z} \times \mathrm{BO} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\tau} & \mathbb{Z} \times \mathrm{BU}. \end{array}$$

Then the spectrum $\Omega(M)$ is homotopy equivalent to the Thom spectrum of the stable vector bundle $-\zeta$ over $\underline{\mathcal{G}}(M)$ defined in the diagram above.

Notice that the fibration $\mathbb{Z} \times \text{BO} \rightarrow \mathbb{Z} \times \text{BU}$ is a principal bundle up to homotopy, with fiber being the infinite loop space U/O . Hence, the spectrum $\Omega(M)$ is homotopy equivalent to a Ω -module spectrum. Now, observe that we have the equivalence, up to homotopy, of U/O -spaces:

$$\underline{\mathcal{G}}(M) \times_{\text{U/O}} \underline{\mathcal{G}}(N) \simeq \underline{\mathcal{G}}(M \times N).$$

This translates to a canonical homotopy equivalence:

$$\mu : \Omega(M) \wedge_{\Omega} \Omega(N) \simeq \Omega(M \times N).$$

Let us now describe the stable symplectic homotopy category $h\mathbb{S}$. The objects of this category will be symplectic manifolds (M, ω) (see remark 2.3), endowed with a compatible almost complex structure.

Definition 2.2. *The spectrum $\Omega(M, N)$ of morphisms in $h\mathbb{S}$ from M to N is the Ω -module spectrum:*

$$\Omega(M, N) := \Omega(\overline{M} \times N).$$

Remark 2.3. *Notice that objects in $h\mathbb{S}$ need not be compact. The price we pay for this, familiar from other contexts, is that we simply lose the identity morphisms for non-compact objects.*

The next step is to define composition. The simplest case

$$\Omega(M, *) \wedge_{\Omega} \Omega(*, N) \rightarrow \Omega(M, N),$$

is the map μ constructed earlier. For the general case, consider $k + 1$ objects M_i with $0 \leq i \leq k$, and let the space $\underline{\mathcal{G}}(\Delta)$ be defined by the pullback:

$$\begin{array}{ccc} \underline{\mathcal{G}}(\Delta) & \xrightarrow{\quad\quad\quad} & \underline{\mathcal{G}}(\overline{M}_0 \times M_1 \times \cdots \times \overline{M}_{k-1} \times M_k) \\ \downarrow \xi & & \downarrow \\ \overline{M}_0 \times (M_1 \times \cdots \times M_{k-1}) \times M_k & \xrightarrow{\Delta} & \overline{M}_0 \times (M_1 \times \overline{M}_1) \times \cdots \times (M_{k-1} \times \overline{M}_{k-1}) \times M_k \end{array}$$

where Δ denotes the product to the diagonals $\Delta : M_i \rightarrow M_i \times \overline{M}_i$, for $0 < i < k$.

Now notice that the fibrations defining the pullback above are direct limits of smooth fibrations with compact fiber. Furthermore, the map Δ is a proper map for any choice of $k + 1$ -objects (even if they are non-compact). In particular, we may construct the Pontrjagin–Thom collapse map along the top horizontal map by defining it as a direct limit of Pontrjagin–Thom collapses for each stage.

Let ζ_i denote the individual structure maps $\underline{\mathcal{G}}(\overline{M}_{i-1} \times M_i) \rightarrow \mathbb{Z} \times \text{BO}$, and let $\eta(\Delta)$ denote the normal bundle of Δ . Performing the Pontrjagin–Thom construction along the top horizontal map in the above diagram yields a morphism of spectra:

$$\varphi : \Omega(M_0, M_1) \wedge_{\Omega} \cdots \wedge_{\Omega} \Omega(M_{k-1}, M_k) \simeq \Omega(\overline{M}_0 \times M_1 \times \cdots \times \overline{M}_{k-1} \times M_k) \rightarrow \underline{\mathcal{G}}(\Delta)^{-\lambda}$$

where $\lambda : \underline{\mathcal{G}}(\Delta) \rightarrow \mathbb{Z} \times \text{BO}$ is the formal difference of the bundle $\bigoplus \zeta_i$ and the pullback bundle $\zeta^* \eta(\Delta)$.

The next step in defining composition is to show that $\underline{\mathcal{G}}(\Delta)^{-\lambda}$ is canonically homotopy equivalent to $\Omega(M_0, M_k) \wedge (M_1 \times \cdots \times M_{k-1})_+$, where $(M_1 \times \cdots \times M_{k-1})_+$ denotes the

manifold $M_1 \times \cdots \times M_{k-1}$ with a disjoint basepoint. To achieve this, it is sufficient to construct a U/O-equivariant map over $\overline{M}_0 \times (M_1 \times \cdots \times M_{k-1}) \times M_k$:

$$\psi : \underline{\mathcal{G}}(\overline{M}_0 \times M_k) \times (M_1 \times \cdots \times M_{k-1}) \longrightarrow \underline{\mathcal{G}}(\Delta),$$

that pulls λ back to the bundle $\zeta \times 0$. The construction of ψ is straightforward. We define:

$$\psi(\lambda, m_1, \dots, m_{k-1}) = \lambda \oplus \Delta(T_{m_1}(M_1)) \oplus \cdots \oplus \Delta(T_{m_{k-1}}(M_{k-1})),$$

where $\Delta(T_m(M)) \subset T_{(m,m)}(M \times \overline{M})$ denotes the diagonal lagrangian subspace. Now let $\pi : \underline{\mathcal{G}}(\Delta)^{-\lambda} \longrightarrow \Omega(M_0, M_k)$ be the projection map that collapses $M_1 \times \cdots \times M_{k-1}$ to a point.

Definition 2.4. We define the composition map to be the induced composite:

$$\pi\varphi : \Omega(M_0, M_1) \wedge_{\Omega} \cdots \wedge_{\Omega} \Omega(M_{k-1}, M_k) \longrightarrow \underline{\mathcal{G}}(\Delta)^{-\lambda} \longrightarrow \Omega(M_0, M_k).$$

We leave it to the reader to check that composition as defined above is homotopy associative.

§2.2 The identity morphism:

We now show that a compact manifold (M, ω) has an identity morphism:

Proposition 2.5. Let M be a compact manifold, and let $[id] : S \longrightarrow \Omega(M, M)$ be a representative of homotopy class of the diagonal (lagrangian) embedding $\Delta : M \longrightarrow \overline{M} \times M$. Then $[id]$ is indeed the identity for the composition defined above.

Proof. Given two manifolds M, N , let $\Delta(M) \subset \overline{M} \times M$ be a diagonal representative of $[id]$ as above. Observe that $\overline{N} \times \Delta(M) \times M$ is transverse to $\overline{N} \times M \times \Delta(M)$ inside $\overline{N} \times M \times \overline{M} \times M$. They intersect along $\overline{N} \times \Delta_3(M)$, where $\Delta_3(M) \subset M \times \overline{M} \times M$ is the triple (thin) diagonal. Hence we get a diagram

$$\begin{array}{ccccc} \Omega(N, M) \wedge S & \longrightarrow & \Omega(N, M) \wedge \Delta(M)^{-\tau} & \xrightarrow{\Delta^{-\tau}} & \Omega(N, M) \wedge_{\Omega} \Omega(M, M) \\ & \searrow & \downarrow & & \downarrow \\ & & \Omega(N, M) & \xrightarrow{=} & \Omega(N, M) \end{array}$$

commutative up to homotopy, where the right vertical map is composition, and the left vertical map is the Pontrjagin–Thom collapse over the inclusion map

$$\overline{N} \times M = \overline{N} \times \Delta_3(M) \longrightarrow \overline{N} \times M \times \Delta(M).$$

Now consider the following factorization of the identity map:

$$\overline{N} \times M = \overline{N} \times \Delta_3(M) \longrightarrow \overline{N} \times M \times \Delta(M) \longrightarrow \overline{N} \times M$$

where the last map is the projection onto the first two factors. Performing the Pontrjagin–Thom collapse over this composite shows that

$$\Omega(N, M) \wedge S \longrightarrow \Omega(N, M) \wedge \Delta(M)^{-\tau} \longrightarrow \Omega(N, M).$$

is the identity. It follows that right multiplication by $[id] : S \rightarrow \Omega(M, M)$ induces the identity map on $\Omega(N, M)$, up to homotopy. A similar argument works for left multiplication. \square

Remark 2.6. Recall that given arbitrary symplectic manifolds M and N , the composition map:

$$\Omega(M, *) \wedge_{\Omega} \Omega(*, N) \simeq \Omega(M, N)$$

induces a natural decomposition of $\Omega(M, N)$. In particular, arbitrary compositions can be canonically factored using this decomposition, and can be computed by applying the “inner product”

$$\Omega(*, N) \wedge_{\Omega} \Omega(N, *) \longrightarrow \Omega.$$

to the factors.

It will also be important below that $h\mathbb{S}$ is a symmetric-monoidal category, with a product given by the cartesian product of symplectic manifolds.

Definition 2.7. The construction of the **oriented** stable symplectic homotopy category $s\mathbb{S}$ is completely analogous, but with O replaced by SO ; the commutative ring spectrum $s\Omega = (U/SO)^{-\zeta}$ defines its coefficients.

Now Ω is an (Eilenberg–MacLane) $H(\mathbb{Z}/2)$ -algebra, so we can regard \mathbb{S} as a category with morphism objects enriched over a classical differential graded algebra. This is not the case for $s\Omega$, but its rationalization $s\Omega \otimes \mathbb{Q}$ is again a generalized Eilenberg–MacLane spectrum, with

$$s\Omega_* \otimes \mathbb{Q} = \Lambda_{\mathbb{Q}}[y_{4i+1}, i > 0]$$

an exterior algebra on certain odd - degree generators. Moreover, the category $s\mathbb{S}$ simplifies considerably when rationalized. In particular, the Thom isomorphism

$$s\Omega(M)_* \otimes \mathbb{Q} \cong H_*(M, s\Omega_* \otimes \mathbb{Q})$$

identifies $s\mathbb{S} \otimes \mathbb{Q}$ with an (Arnol’d-Hörmander-Maslov...) category of symplectic manifolds, whose morphisms are classical cohomological correspondences with compact support, but with coefficients in the graded ring $s\Omega_* \otimes \mathbb{Q} \cong H_*(U/SO_+, \mathbb{Q})$.

3. THE WALDHAUSEN K-THEORY OF $s\Omega$ AND A SYMPLECTIC INVARIANT.

In this section we will study the coefficient spectrum $s\Omega$ through its Waldhausen K-theory. The spectrum $s\Omega$ is a connective spectrum with $\pi_0(s\Omega) = \mathbb{Z}$. Let us consider the fibration:

$$K(\pi) \longrightarrow K(s\Omega) \longrightarrow K(\mathbb{Z})$$

where π is the fiber of the zero-th Postnikov section: $s\Omega \rightarrow H\mathbb{Z}$.

Proposition 3.1. Let $\overline{K}(s\Omega)$ denote the cofiber of the canonical map $K(S) \rightarrow K(s\Omega)$. Then rationally, the spectrum $K(\pi)$ is equivalent to $\overline{K}(s\Omega)$. In particular the above fibration admits a canonical rational splitting, and there exist polynomial classes y_{4i+2} in degree $4i+2$ such that $\pi_*K(\pi)$ is isomorphic to the augmentation ideal:

$$\pi_*K(\pi) \otimes \mathbb{Q} = \mathbb{Q}[y_{4i+2}]_{>0}.$$

Furthermore, rationally $\pi_*K(\pi)$ can be identified with the (injective) image in homotopy of the group completion map:

$$\Omega^{\infty}\Sigma^{\infty}(B(U/SO)) \longrightarrow BGL_{\infty}(s\Omega)^+ = \Omega^{\infty}K(s\Omega).$$

Proof. Since $K(S)$ is rationally equivalent to $K(\mathbb{Z})$, the first part of the claim is clear. Via the Thom isomorphism, we may identify $s\Omega$ rationally with the ring spectrum $\Sigma^\infty(U/SO)_+$. In particular $\pi_* s\Omega \otimes \mathbb{Q} = \Lambda(y_{4i+1})$. Now invoking results from [3], we see that $\pi_* K(\pi)$ can be identified with the set of positive degree elements in $\mathrm{THH}_*(U/SO_+)$ in the kernel of the Connes boundary operator. These elements are given by the augmentation ideal in the polynomial algebra $H_*(B(U/SO)) = \mathbb{Q}[y_{4i+2}]$. The classes y_{4i+2} (ancestors of the exterior classes y_{4i+1}) are detected in rational homotopy along the inclusion given by: $B(U/SO) \rightarrow \mathrm{BGL}_\infty(\Sigma^\infty(U/SO)_+)^+$. The result now follows. \square

We now fix a compact symplectic manifold (M, ω) , and describe an invariant of the classifying space of the symplectomorphism group: $\mathrm{BSymp}(M)$, with values in $K(s\Omega)$. Recall from [27] that one has a map:

$$\gamma : \mathrm{BSymp}(M) \rightarrow \mathrm{BGL}(s\Omega(M, M)),$$

where the group of units $\mathrm{GL}(s\Omega(M, M)) = \mathrm{Aut}_{s\Omega}(s\Omega(M))$ is defined as the components that induce invertible $\pi_0(s\Omega)$ -module maps in homotopy. The map γ is defined by de-looping the map that sends a symplectomorphism φ to its graph in $\overline{M} \times M$. Recall [27], that the map γ was explicitly constructed as a map that classifies a parametrized bundle of $s\Omega$ -module spectra over $\mathrm{BSymp}(M)$ with fiber $s\Omega(M)$. This bundle was obtained as a fiberwise compactification of the formal negative of a bundle $J(\zeta)$ defined over a space $\underline{\mathcal{G}}(J(M))$ fibering over $\mathrm{BSymp}(M)$:

$$\begin{array}{ccccc} \mathrm{BSymp}(M) & \longleftarrow & \underline{\mathcal{G}}(J(M)) & \xrightarrow{J(\zeta)} & \mathbb{Z} \times \mathrm{BO} \\ \downarrow = & & \downarrow \xi & & \downarrow \\ \mathrm{BSymp}(M) & \longleftarrow & J(M) & \xrightarrow{J(\tau)} & \mathbb{Z} \times \mathrm{BU}, \end{array}$$

where $J(M) \rightarrow \mathrm{BSymp}(M)$ is equivalent to the universal fiber bundle with fiber M and structure group $\mathrm{Symp}(M)$.

Definition 3.2. Since $s\Omega(M)$ is a finite cellular $s\Omega$ -module, we may define a stabilization map:

$$Q(M) : \mathrm{BGL}(s\Omega(M, M)) \rightarrow \Omega^\infty K(s\Omega)$$

where $K(s\Omega)$ denotes Waldhausen's K -theory spectrum of the commutative ring spectrum $s\Omega$. The $K(s\Omega)$ -valued parametrized index is defined as the composite:

$$I(M) = Q(M) \circ \gamma : \mathrm{BSymp}(M) \rightarrow \Omega^\infty K(s\Omega).$$

Note that such a stabilization can be defined for the category \mathbb{S} as well.

Theorem 3.3. In cohomology, the parametrized index map $I(M)$ can be identified with the map given by applying the Becker–Gottlieb transfer: $[tr] : H^*(J(M)) \rightarrow H^*(\mathrm{BSymp}(M))$ to a subalgebra inside $H^*(J(M))$. This subalgebra is generated by the odd Newton polynomials (see discussion before 4.5) in the Chern classes of the fiberwise tangent bundle $J(M) \rightarrow \mathrm{BSymp}(M)$.

Proof. By [21] (Thm 8.5), the map $I(M)$ has a factorization:

$$\mathrm{BSymp}(M) \rightarrow \Omega^\infty \Sigma^\infty(J(M)_+) \rightarrow \Omega^\infty A(J(M)) \rightarrow \Omega^\infty K(s\Omega)$$

where $A(J(M))$ is the Waldhausen K -theory of $J(M)$, the first map is the Becker–Gottlieb transfer, the second is the natural transformation from $\Sigma^\infty(X_+)$ to $A(X)$, and the final map is the one that classifies the stable bundle of $s\Omega$ -spectra over $J(M)$.

Now let $\mathrm{BGl}(s\Omega)$ denote the classifying space of the group of units of $s\Omega$ [5], which classifies bundles of parametrized rank one $s\Omega$ -module spectra. Since $s\Omega$ is an E_∞ -ring spectrum, $\mathrm{BGl}(s\Omega)$ is an infinite loop space, with product defined by the fiberwise smash product of parametrized $s\Omega$ -module spectra.

The projection map $U \rightarrow U/\mathrm{SO}$ is equivalent to a map of infinite loop spaces, and the bundle ζ over U/SO (that defines $s\Omega = (U/\mathrm{SO})^{-\zeta}$ as a Thom spectrum) restricts to the trivial bundle over U . Consequently we have a left-action map: $U_+ \wedge (U/\mathrm{O})^{-\zeta} \rightarrow (U/\mathrm{O})^{-\zeta}$. This translates to a map of H-spaces $\mathrm{BU} \rightarrow \mathrm{BGl}(s\Omega)$.

Using naturality, we have the following description of $I(M)$ given by the commutative diagram with the vertical maps induced by the map that classifies the (rank one) bundle of $s\Omega$ -spectra over $J(M)$:

$$\begin{array}{ccccccc}
\Sigma^\infty(\mathrm{BSymp}(M)_+) & \xrightarrow{tr} & \Sigma^\infty(J(M)_+) & \longrightarrow & A(J(M)) & \longrightarrow & K(s\Omega) \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow = \\
\Sigma^\infty(\mathrm{BSymp}(M)_+) & \longrightarrow & \Sigma^\infty(\mathrm{BU}_+) & \longrightarrow & A(\mathrm{BU}) & \longrightarrow & K(s\Omega) \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow = \\
\Sigma^\infty(\mathrm{BSymp}(M)_+) & \longrightarrow & \Sigma^\infty(\mathrm{BGl}(s\Omega)_+) & \longrightarrow & A(\mathrm{BGl}(s\Omega)) & \longrightarrow & K(s\Omega)
\end{array}$$

Now recall from the proof of claim 3.1, that the map $\mathrm{BU} \rightarrow \Omega^\infty K(s\Omega)$ is surjective onto the vector space spanned by the generators y_{4k+2} in rational homotopy. It follows that in rational cohomology, the map: $H^*(\Omega^\infty K(s\Omega)) \rightarrow H^*(\mathrm{BU})$ is surjective onto the polynomial algebra generated by the odd Newton polynomials $N_{2k+1}(c_1, c_2, \dots, c_{2k+1})$ in the universal Chern classes. Invoking the above diagram, the proof follows. \square

Remark 3.4. Proposition 3.1 shows that we have a rational pullback diagram:

$$\begin{array}{ccc}
\Sigma^\infty \mathrm{B}(U/\mathrm{SO})_+ & \longrightarrow & S \\
\downarrow & & \downarrow \\
K(s\Omega) & \longrightarrow & K(\mathbb{Z})
\end{array}$$

Hence we may define secondary rational (Reidemeister) invariants in dimensions $4k$, for families of symplectic manifolds (M, ω) that admit a prescribed null homotopy for the parametrized index.

4. A CANONICAL REPRESENTATION

The next item on our agenda is a canonical representation of $h\mathbb{S}$. For the applications we have in mind, $hs\mathbb{S}$ will be the relevant category, but for the moment we proceed with constructions which work generally.

Given a symplectic manifold (M, ω) recall that the morphism spectrum $\Omega(*, M)$ can be identified with the Ω -module spectrum $\Omega(M)$. In particular, right composition in $h\mathbb{S}$ yields a representation:

$$\mathcal{F} : h\mathbb{S} \rightarrow \Omega\mathcal{S}, \quad \mathcal{F}(M) = \Omega(M),$$

where $\Omega\mathcal{S}$ denotes the homotopy category of Ω -module spectra. Recall that the category $h\mathcal{S}$ is a symmetric-monoidal category, with the monoidal structure given by the cartesian product of symplectic manifolds. Since $\Omega(M \times N)$ is equivalent to $\Omega(M) \wedge_{\Omega} \Omega(N)$, the functor \mathcal{F} is monoidal. This functor has a motivic group $\text{Aut}_{\wedge}(\Omega)$ of natural A_{∞} Ω -module equivalences with itself. This maps to a group G_{Ω} of monoidal automorphisms of the functor

$$\mathcal{F}_E : E \mapsto [M \mapsto \Omega(M) \wedge_{\Omega} E]$$

on the category of Ω -module spectra. The analogous construction in $hs\mathcal{S}$ is nontrivial when tensored with \mathbb{Q} , which has the advantage of allowing us to work with Lie algebras rather than the groups themselves.

§4.1 The Lie algebra of primitive automorphisms:

Given a symplectic manifold M , recall that $\Omega(M)$ was defined as a Thom spectrum: $\underline{\mathcal{G}}(M)^{-\zeta}$. Here $\pi : \underline{\mathcal{G}}(M) \rightarrow M$ was a principal U/O -bundle, supporting a stable real vector bundle $\zeta : \underline{\mathcal{G}}(M) \rightarrow \text{BO}$ of virtual dimension m . The bundle π is classified by the map:

$$\tau(\Omega, M) : M \rightarrow \text{B}(U/O).$$

Theorem 4.1. *Given an object of $h\mathcal{S}$ represented by a symplectic manifold (M, ω) , then $\tau(M, \Omega)$ factors through the map $\tau(M) : M \rightarrow \text{BU}$ that classifies the tangent bundle of M , followed by the projection map $\text{BU} \rightarrow \text{B}(U/O)$. Furthermore, the restriction of $\tau(\Omega, M \times \overline{M})$ along the diagonal: $\Delta : M \rightarrow M \times \overline{M}$ is trivial.*

Proof. Notice that the bundle $\pi : \underline{\mathcal{G}}(M) \rightarrow M$ was induced from the frame bundle of M , with structure group U , along the left action of U on U/O , so we can lift the map $\tau(\Omega, M)$, to BU , to get the factorization through $\tau(M)$. This proves the first part of the claim. Now notice that the restriction of the tangent bundle of $M \times \overline{M}$ along the diagonal Δ is canonically isomorphic to the complexification of the tangent bundle of M . This may be restated as saying that one has a unique lift $\tau(\Delta)$ that makes the following commute:

$$\begin{array}{ccc} M & \xrightarrow{\tau(\Delta)} & \text{BO} \\ \downarrow \Delta & & \downarrow \\ M \times \overline{M} & \xrightarrow{\tau} & \text{BU}. \end{array}$$

It follows that $\tau(\Omega, M \times \overline{M})$ is trivial when restricted along Δ . □

Remark 4.2. *The projection map $\text{BU} \rightarrow \text{B}(U/O)$ is injective in cohomology away from two. It is easy to see that its image is generated by classes d_{2i+1} in degree $4i + 2$, whose generating function can be expressed in terms of the Chern classes as:*

$$\sum_i d_i = \frac{\sum_i (-1)^i c_i}{\sum_i c_i} \equiv 1 - \sum_i 2c_{2i+1} + \text{decomposables}.$$

Similarly, if one considers the inclusion $\text{BO} \rightarrow \text{BU}$, then this map is injective in homology away from two. If we let $\sum_i b_i$ denote the homogeneous generators in the image of $H_(\mathbb{CP}^{\infty}) \subset H_*(\text{BU})$, then the image of $H_*(\text{BO})$ is the sub (polynomial) algebra generated by classes a_i (away from two), given by:*

$$\sum_i a_i = \left(\sum_i b_i \right) \left(\sum_i (-1)^i b_i \right) \equiv 1 + \sum_{i>0} 2b_{2i} + \text{decomposables}.$$

Definition 4.3. Henceforth, we work in the oriented category $hs\mathbb{S}$, and we assume that $\pi_* E$ is a \mathbb{Q} -vector space. Define $\mathcal{P}_E(B(U/SO))$ to be the graded E^* -submodule of $\tilde{E}^*(B(U/SO))$ consisting of primitive elements in the (commutative) Hopf algebra $E^*(B(U/SO))$.

Theorem 4.4. $\mathcal{P}_E(B(U/SO))$ acts on \mathcal{F}_E by graded natural transformations. In other words, there is a natural map of graded E^* -modules:

$$\mathcal{P}(E) : \mathcal{P}_E(B(U/SO)) \longrightarrow \text{End}(\mathcal{F}_E).$$

Furthermore, the image of $\mathcal{P}(E)$ is contained in the subgroup of primitive natural transformations, defined as natural transformations φ that are additive with respect to the monoidal structure:

$$\varphi(X \wedge_E Y) = \varphi(X) \wedge_E Y + X \wedge_E \varphi(Y).$$

Proof. Fix an object (M, ω) of $hs\mathbb{S}$. Given an element $\alpha \in \mathcal{P}_E(B(U/SO))$, we define the action of $\mathcal{P}(E)(\alpha)$ on $\mathcal{F}_E(M)$ as the cap product with α , described as the composite map:

$$\alpha_* : s\Omega(M)_E \longrightarrow s\Omega(M)_E \wedge M_+ \longrightarrow s\Omega(M)_E \wedge B(U/SO)_+ \longrightarrow s\Omega(M)_E,$$

where the first map is induced by the diagonal map $s\mathcal{G}(M)^{-\zeta} \longrightarrow s\mathcal{G}(M)^{-\zeta} \wedge M_+$. The second map is induced by $\tau(s\Omega, M)$, and the third map above is given by capping with the class α . [The reader should bear in mind that $\alpha_* = \alpha(M)_*$ depends on M .] The construction above defines a map of E^* -modules:

$$\mathcal{P}(E) : \mathcal{P}_E(B(U/SO)) \longrightarrow \text{End}_E(\mathcal{F}_E(M)).$$

of E^* -modules. It remains to show that $\mathcal{P}(E)$ yields primitive natural transformations. Consider symplectic manifolds M and N . Recall that $s\mathcal{G}(M \times N)$ is equivalent to the external product bundle $s\mathcal{G}(M) \times_{U/SO} s\mathcal{G}(N)$. In particular, the element $\tau(s\Omega, M \times N)$ decomposes as the composite:

$$M \times N \longrightarrow B(U/SO) \times B(U/SO) \longrightarrow B(U/SO).$$

Given a primitive class $\alpha \in \mathcal{P}_E(B(U/SO))$, the pullback of α along $\tau(s\Omega, M \times N)$ is therefore given by $\alpha(M)_* \wedge 1 + 1 \wedge \alpha(N)_*$. This is exactly the definition of a primitive endomorphism.

To see that $\mathcal{P}(E)(\alpha)$ is a natural transformation, we need to show that the diagram

$$\begin{array}{ccc} s\Omega(M)_E \wedge_E s\Omega(M, N)_E & \xrightarrow{\alpha_* \wedge \alpha_*} & s\Omega(M)_E \wedge_E s\Omega(M, N)_E \\ \downarrow & & \downarrow \\ s\Omega(N)_E & \xrightarrow{\alpha_*} & s\Omega(N)_E, \end{array}$$

commutes; where the vertical maps are induced by composition in $h\mathbb{S}$, and the top horizontal map: $\alpha_* \wedge \alpha_* : s\Omega(M)_E \wedge_E s\Omega(M, N)_E \longrightarrow s\Omega(M)_E \wedge_E s\Omega(M, N)_E$ denotes the external smash product of the two maps $\alpha(M)_*$ and $\alpha(\overline{M} \times N)_*$.

By the primitivity of α_* , we write $\alpha(\overline{M} \times N)_*$ as the sum $\alpha(\overline{M})_* \wedge 1 + 1 \wedge \alpha(N)_*$. This decomposition allows us to reduce the general case to the special case when N is a point. In other words, we would like to show that the following special case of the above diagram

commutes:

$$\begin{array}{ccc}
s\Omega(*, M)_E \wedge_E s\Omega(M, *)_E & \xrightarrow{\alpha_* \wedge \alpha_*} & s\Omega(*, M)_E \wedge_E s\Omega(M, *)_E \\
\downarrow & & \downarrow \\
E & \xrightarrow{0} & E.
\end{array}$$

To show this, recall that the composition $s\Omega(*, M)_E \wedge_E s\Omega(M, *)_E \rightarrow E$ is obtained by restricting along the diagonal $\Delta : M \rightarrow M \times \overline{M}$. By claim 4.1, we see that the restriction of $\tau(s\Omega, M \times \overline{M})$ along Δ is trivial. It follows that the $\alpha_* \wedge \alpha_*$ followed by composition is trivial. \square

§4.2 The image of $\mathcal{P}(E)$:

Let $\mathcal{P}_E(\text{BU}) \subset \tilde{E}^*(\text{BU})$ denote the submodule of primitives. For complex oriented theories it is a standard fact that this is a free (completed) E^* -module generated by the Newton polynomials $N_k(c_1, \dots, c_k)$, in the Chern classes. These Newton polynomials $N_i(\sigma_1, \dots, \sigma_i)$ are defined by writing the power symmetric functions $x_1^i + x_2^i + \dots + x_i^i$ in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \dots, \sigma_i$. Rationally, the classes N_k can be expressed in terms of the Chern classes c_k , or the classes d_k of remark 4.2, by comparing the homogeneous terms in the formal graded equalities (see [31] Ch.1):

$$\sum_{k \geq 0} c_k = \prod_{i \geq 0} \exp \frac{(-1)^i N_{i+1}}{i+1}, \quad \sum_{k \geq 0} d_k = \prod_{i \geq 0} \exp \frac{-2N_{2i+1}}{2i+1}.$$

Up to a scaling factor of $k!$, $N_k(c_1, \dots, c_k)$ is the homogeneous degree $2k$ term in the Chern character ch_k for the universal virtual vector bundle over BU. Notice that any theory E such that E_* is a \mathbb{Q} -vector space, is complex orientable.

Theorem 4.5. *Assume that E_* is a \mathbb{Q} -vector space. Fix a complex orientation on E (see remark 4.7 below). Then, in cohomology, the map induced by the projection:*

$$\mathcal{P}_E(\text{B}(\text{U}/\text{SO})) \rightarrow \mathcal{P}_E(\text{BU})$$

is injective onto the free (completed) E^ -module generated by the primitives $N_{2k+1}(c_1, \dots, c_{2k+1})$ in degree $4k+2$, with $k \geq 0$ (see [32]). In particular, the image of $\mathcal{P}(E)$ is generated by operators acting on $s\Omega(M)$ via multiplication with the classes $ch_{2k+1}(\tau)$ (compare with [30] Theorem 9).*

Proof. Since E_* is a \mathbb{Q} -vector space, we may assume E is a generalized Eilenberg–MacLane spectrum. Notice that the projection map $\text{BU} \rightarrow \text{B}(\text{U}/\text{SO})$ is a map of H-spaces, and in homology, it maps the indecomposable elements in degrees $4k+2$ isomorphically onto the indecomposables in the Hopf-algebra $E_*(\text{B}(\text{U}/\text{SO}))$ (see remark 4.2). Dually, it follows that $\mathcal{P}_E(\text{B}(\text{U}/\text{SO}))$ maps isomorphically to the completed subspace generated by primitives in degree $4k+2$ in $\mathcal{P}_E(\text{BU})$. By the above discussion, this is the completed subspace generated by the odd Newton polynomials in the Chern classes. \square

Corollary 4.6. *Let $G(E)$ denote the (pro) abelian group generated by the formal exponentials of the form $\text{Exp}(t ch_{2k+1}(\tau))$, with t being any homogeneous element of degree $4k+2$ in the \mathbb{Q} -vector space E_* , where $k \geq 0$. Then $G(E)$ acts by degree-preserving monoidal automorphisms on the functor \mathcal{F}_E . In particular, $G(E)$ is a subgroup of the motivic Galois group. Presently, we will describe a canonical integral form for this group.*

Remark 4.7. *The reader may wish to verify that the completed subspace generated by the odd Newton polynomials in the Chern classes is independent of the choice of complex orientation. In particular, the same holds for the group $G(E)$.*

§4.3 An integral candidate for the (abelianized) Grothendieck–Teichmüller group:

In theorems 4.4 and 4.5 we described the Lie algebra of the group $G(E)$ as being the primitives in $\tilde{E}^*(B(U/SO))$. This implies that the cotangent space of $G(E)$ at the identity element should be interpreted as the vector space dual to these primitives. This dual space can be canonically identified with a subspace of the indecomposables: $Q(E_*(B(U/SO))) = I/I^2$, where I is the augmentation ideal in $E_*(B(U/SO))$. This suggests that one must think of the commutative ring spectrum $s\Omega \wedge B(U/SO)_+$ as “functions on a derived avatar” of the abelianized Grothendieck–Teichmüller group.

The spectrum $s\Omega \wedge B(U/SO)_+$ can be constructed functorially from $s\Omega$: Indeed, we know by [8] (Prop. 7.3), that $s\Omega \wedge B(U/SO)_+$ is equivalent to $\mathrm{THH}(s\Omega)$ as commutative algebras. Notice in fact, that $s\Omega \wedge B(U/SO)_+$ is a commutative Hopf-algebra spectrum in the category of $s\Omega$ -module spectra [3]. The above discussion leads naturally to the:

Definition 4.8. *Define a derived group scheme $\mathbf{G} = \mathrm{Spec} \mathrm{THH}(s\Omega)$, whose E -points for an arbitrary commutative $s\Omega$ -algebra E is defined to be the group of homotopy classes of $s\Omega$ -algebra maps from $\mathrm{THH}(s\Omega)$ to E :*

$$\mathbf{G}(E) = \mathrm{Alg}_{s\Omega}(\mathrm{THH}(s\Omega), E) = \mathrm{Alg}_S(B(U/SO)_+, E).$$

Theorem 4.9. *Given a commutative $s\Omega$ -algebra E , the group $\mathbf{G}(E)$ acts by monoidal automorphisms of the functor \mathcal{F}_E .*

Proof. The proof of this theorem is similar to the proof of theorem 4.4. As before, given a point in $\beta \in \mathbf{G}(E)$, we get an automorphism of $\mathcal{F}_E(M)$ given by the cap product with β :

$$\beta_* : s\Omega(M)_E \longrightarrow s\Omega(M)_E \wedge M_+ \longrightarrow s\Omega(M)_E \wedge B(U/SO)_+ \longrightarrow s\Omega(M)_E.$$

Since β is a map of algebras, we see that β_* preserves the monoidal structure of \mathcal{F}_E . So the only thing left to check is that β_* is a natural transformation. As in the proof of theorem 4.4, we may reduce this question to showing that the following diagram commutes:

$$\begin{array}{ccc} s\Omega(*, M)_E \wedge_E s\Omega(M, *)_E & \xrightarrow{\beta_* \wedge \beta_*} & s\Omega(*, M)_E \wedge_E s\Omega(M, *)_E \\ \downarrow & & \downarrow \\ E & \xrightarrow{=} & E. \end{array}$$

Again, as in the proof of 4.4, this requires showing that the restriction of $\tau(s\Omega, M \times \overline{M})$ along Δ below, factors through the unit of E :

$$M_+ \longrightarrow M_+ \wedge \overline{M}_+ \longrightarrow B(U/SO)_+ \longrightarrow E.$$

But this follows from theorem 4.1, and the fact that β is a ring map. □

In the above interpretation, the action of \mathbf{G} on $s\Omega(M)$ is induced by the following coaction:

$$s\Omega(M) \longrightarrow s\Omega(M) \wedge_{s\Omega} \mathrm{THH}(s\Omega) = s\Omega(M) \wedge B(U/SO)_+.$$

Remark 4.10. Recall the stabilization map $BU \rightarrow K(s\Omega)$ defined in the proof of theorem 3.3. Composing this with the Dennis trace map, we get a map $BU \rightarrow THH(s\Omega)$. One may identify this composite with the map induced by the inclusion of the unit $S \rightarrow s\Omega$:

$$BU \rightarrow B(U/SO)_+ \rightarrow s\Omega \wedge B(U/SO)_+ = THH(s\Omega).$$

This suggests a close relation between the Waldhausen K -theory of $s\Omega$ and the pro-abelian group-scheme \mathbf{G} . This is strikingly reminiscent of Kato's higher classfield theory [26][Theorem 2.1], which relates the algebraic K -theory of higher local fields to the Galois groups of their abelian extensions.

§4.4 Final remarks and speculation

In the sections that follow, we borrow notation from [32], where $S[G_+]$ denotes the suspension spectrum of G , viewed as a kind of group ring, for an H -space G .

Kontsevich's 1999 paper suggests that an action of $GT_{\mathbb{Q}}$ defines a deformation of the complexified \hat{A} -genus, which can be interpreted as associated to the formal group law with $\Gamma(x)^{-1}$ as its exponential (where $\Gamma(x)$ denotes the Gamma-function, [32][§2.3.1]):

We seek to generalize Kontsevich's result to our context. Let $MU \rightarrow MSO$ denote the forgetful map, and let $MU \rightarrow S[BU_+] \wedge MU$ denote the diagonal map. Then given a commutative ring spectrum E with an SO -orientation $\rho : MSO \rightarrow E$, we have a map of ring spectra Γ_ρ :

$$\Gamma_\rho : MU \rightarrow S[BU_+] \wedge MU \rightarrow S[B(U/SO)_+] \wedge MSO \rightarrow S[B(U/SO)_+] \wedge E,$$

Note that the map Γ_ρ above can be expressed as a morphism:

$$\Gamma_\rho : MU \rightarrow THH(s\Omega) \wedge_{s\Omega} E.$$

Using this description, one may generalize Kontsevich's construction in our framework: namely, one may define a torsor of deformations of the ρ -orientation under the action of the group $\mathbf{G}(E)$ as follows: Given an element in $\mathbf{G}(E)$ represented by a map of ring spectra $\beta : B(U/SO)_+ \rightarrow E$, the corresponding deformation of ρ is given by capping β with Γ_ρ :

$$\rho_\beta = \beta \cap \Gamma_\rho : MU \rightarrow S[B(U/SO)_+] \wedge E \rightarrow E.$$

There is a metaplectic analog of this whole picture. Recall ([27], §8), that the unitary group U admits a natural double cover \tilde{U} that supports the square-root of the determinant homomorphism. The forgetful map $U \rightarrow SO$ lifts to a unique map $\tilde{U} \rightarrow \text{Spin}$. Therefore, given a spectrum E with a Spin -orientation $\rho : M\text{Spin} \rightarrow E$, we have the corresponding:

$$\tilde{\Gamma}_\rho : M\tilde{U} \rightarrow THH(s\tilde{\Omega}) \wedge_{s\tilde{\Omega}} E,$$

leading to a $\mathbf{G}(E)$ -torsor of deformations of ρ as before.

Remark 4.11. Notice that the maps:

$$M\tilde{U} \rightarrow S[B(\tilde{U}/\text{Spin})_+] \wedge M\text{Spin}, \quad MU \rightarrow S[B(U/SO)_+] \wedge MSO$$

are equivalences away from the prime two. In addition, the spaces $B(\tilde{U}/\text{Spin})$, $B(U/SO)$ and (Sp/U) are also equivalent away from two.

An example of such a deformation is the one suggested by Kontsevich above: Let E denote complex K-theory with complex coefficients: $KU \otimes \mathbb{C}$ supporting the \hat{A} -orientation (or rather, its scalar extension by \mathbb{C}): $\hat{A} : MSpin \rightarrow KU \otimes \mathbb{C}$. In [32][§2.3.1]), we construct a deformation \hat{A}_ζ by specializing the generators of the polynomial algebra $H_*(Sp/U, \mathbb{Q})$ to odd zeta-values in $KU \otimes \mathbb{C}$, graded using suitable powers of the Bott class. The genus associated to \hat{A}_ζ is therefore given by the composite:

$$\hat{A}_\zeta : MU_* \rightarrow H_*(Sp/U_+, KU \otimes \mathbb{Q}) \rightarrow KU_* \otimes \mathbb{C}$$

which restricts to the \hat{A} -genus on MSp (which is MSO , or $MSpin$ away from two), and sends manifolds of dimension $\equiv 2 \pmod{4}$ to $i\mathbb{R} \subset \mathbb{C}$; more precisely, the primitives of $H_*(Sp/U_+, \mathbb{Q})$, interpreted as symmetric functions, are sent (as explained in the appendix below) to odd zeta-values, graded using the Bott class.

Let us now describe the geometry behind this deformation induced by the motivic-Galois group $G(E)$. First consider the spectrum $B(U/SO)_+ \wedge MSO$. Using the language used in Section §2, we may identify an element in $\pi_k(B(U/SO)_+ \wedge MSO) = \pi_k(THH(s\Omega) \wedge_{s\Omega} MSO)$ is the stabilization (with respect to n) of the data given by a concordance class of oriented manifolds $M^{k+n} \subset \mathbb{R}^\infty \times \mathbb{R}^n$, that are proper over \mathbb{R}^n and are endowed with a principal bundle of oriented lagrangian grassmannians classified by a map $\theta : M^{k+n} \rightarrow B(U/SO)$. In this language, the map $\Gamma_\rho : \pi_k(MU) \rightarrow \pi_k(THH(s\Omega) \wedge_{s\Omega} MSO)$ described above identifies a stably almost complex manifold M^k with the underlying oriented manifold, endowed with the formal negative of the bundle ξ , where $\xi : s\mathcal{G}(M) \rightarrow M$ is the bundle of oriented lagrangians in the tangent bundle (see §2).

Given an E -valued genus ρ , the action of $G(E)$, in the language used above, corresponds to deforming ρ along different choices of multiplicative E -theory characteristic classes of the bundle θ . The analogous statements hold in the metaplectic case.

Remark 4.12. *It is a compelling question to ask about the meaning of these (lagrangian-bundle) deformations in terms of elliptic differential operators (like the Dirac operator) that define genera. This framework bears a striking resemblance to the construction of the analytic torsion classes [11] and one would like to know if there is any relation.*

5. Appendix: GROTHENDIECK-TEICHMÜLLER GROUPS

We recall some of the complex history [1][§25] of this subject, which has deep connections to homotopy theory but which may be unfamiliar.

§5.1 The basic definition(s)

For topologists, the operad defined by Artin's braid groups is a good place to start: it has few automorphisms, but its completions are less rigid. Ihara has studied its system of profinite completions, but for our purposes Drinfeld's work [20] on the automorphism group of its system of Malcev \mathbb{Q} -completions will be more relevant. Kontsevich compared the latter object to the group of homotopy automorphisms of the rational chains on the little disk operad, and suggested [30][§4.4] that both these objects are isomorphic to the motivic Galois group of a certain [19] Tannakian category of mixed Tate motives.

We follow Kontsevich's lead here, and refer to all these conjecturally equivalent proalgebraic \mathbb{Q} -groupschemes as **the** Grothendieck-Teichmüller group $\mathrm{GT}_{\mathbb{Q}}$; we take it to be an extension of the form:

$$1 \rightarrow \mathfrak{F} \longrightarrow \mathrm{GT}_{\mathbb{Q}} \longrightarrow \mathbb{G}_m \rightarrow 1$$

with \mathfrak{F} pro-unipotent, defined by a graded Lie algebra \mathfrak{f} free on generators z_{2i+1} , $i > 0$, associated to the generators of the rank one abelian groups $K_{4n+1}^{\mathrm{alg}}(\mathbb{Z})$ through the manifestation of $\mathrm{GT}_{\mathbb{Q}}$ as the motivic group of a \mathbb{Q} -linear category $\mathrm{MTM}_{\mathbb{Z}}$ of mixed Tate motives, generated by certain cell-like objects $\mathbb{Z}(n)$ satisfying:

$$\begin{aligned} \mathrm{Ext}_{\mathrm{MTM}}^*(\mathbb{Z}(0), \mathbb{Z}(n)) &= \mathbb{Q} \text{ if } * = 0 \text{ and } n = 0 \\ &= K_{4n+1}^{\mathrm{alg}}(\mathbb{Z}) \otimes \mathbb{Q} \text{ if } * = 1, \end{aligned}$$

and is zero otherwise. Borel's work on regulators thus relates the generators z_{2i+1} to the (conjecturally transcendental) zeta-values $\zeta(2i+1)$.

The affine groupscheme \mathfrak{F} has a \mathbb{G}_m -action, and the associated commutative Hopf algebra of functions $\mathrm{QSymm} \otimes \mathbb{Q}$ is dual to the universal enveloping algebra of its Lie algebra \mathfrak{f} . That is the free associative \mathbb{Q} -algebra:

$$U(\mathfrak{f}) = \mathbb{Q}\langle\langle z_{2i+1} \mid i > 0 \rangle\rangle = \mathrm{NSymm} \otimes \mathbb{Q}$$

of rational noncommutative symmetric functions [6, 15, 23], with diagonal defined by the juxtaposition coproduct. The (self-dual) Hopf algebra Symm of classical symmetric functions is dual to the abelianization of NSymm : the abelianization $\mathfrak{f} \rightarrow \mathfrak{f}_{\mathrm{ab}}$ defines a homomorphism $U(\mathfrak{f}) \rightarrow U(\mathfrak{f}_{\mathrm{ab}})$ dual to the inclusion of the symmetric functions in the quasisymmetric ones.

Now consider a map that sends the Newton's power sums:

$$N_n := \sum_{k \geq 1} x_k^n \in \mathrm{Symm} \subset \mathrm{QSymm}$$

to the real number given by the Riemann-zeta value: $\zeta(n)$, under the specialization that sends $x_k \mapsto 1/k$; this extends from Symm to a ring homomorphism:

$$\zeta : \mathrm{QSymm} \longrightarrow \mathrm{MZV} \subset \mathbb{R}$$

with values in a certain graded algebra [7, 22] of real multizeta values³.

Remarkably enough, these multizeta values play an important role in Connes, Kreimer, and Marcolli's Galois-theoretic reinterpretation [16, 17] of the classical BPS renormalization theory of Feynman integrals, in which MZVs appear ubiquitously in explicit computations. Kontsevich found an action [30][§4.6 Th 9] of the abelianization of $\mathrm{GT}_{\mathbb{Q}}$ on a moduli space for deformation quantizations of Poisson manifolds, through an action of the little disk operad on Hochschild homology. These developments led Cartier [14] to suggest that $\mathrm{GT}_{\mathbb{Q}}$ is in some sense a *cosmic* Galois group.

In this appendix, we sketch the construction of a homotopy-theoretic analog GT of $\mathrm{GT}_{\mathbb{Q}}$, and define a morphism:

$$\mathrm{GT} \longrightarrow \mathrm{G}$$

³The case $n = 1$, ie $i = 0$, which needs some care [13], is interestingly absent from the constructions considered above.

which rationalizes to the abelianization defined above. The underlying idea [33, 34] is the possible existence of a category of motives [10] over the sphere spectrum S (rather than over the integers \mathbb{Z}), whose morphism objects are modules over Waldhausen's $K(S)$ – which is rationally isomorphic to $K(\mathbb{Z})$.

§5.2 Implications of Koszul duality

For any spectrum X , stabilization defines a morphism:

$$S[\Omega^\infty X_+] \longrightarrow S \vee X$$

of ring spectra, where the right-hand side is interpreted as a square-zero extension of the sphere spectrum S . The induced map:

$$\text{Symm}(H_*(X, \mathbb{Q})) \longrightarrow \mathbb{Q} \oplus H_*(X, \mathbb{Q})$$

on rational homology is thus the quotient of the free graded-commutative algebra on the left by the square of its augmentation ideal. If for example $X = \Sigma kO$, then (away from two) $S[\Omega^\infty X_+]$ is equivalent to the suspension spectrum of U/SO , whose homology is an exterior algebra. The resulting map:

$$\text{Symm} = H_*(B(U/SO), \mathbb{Q}) = \text{Tor}_{H_*(U/SO, \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \longrightarrow \text{Tor}_{H_*(S \vee \Sigma kO, \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}\text{Symm}$$

is then (by old work of Tate on homology of local rings) the inclusion of a polynomial (symmetric) Hopf algebra on a certain module of indecomposables, into the corresponding algebra of quasisymmetric functions (on Lyndon words in those indecomposables). The resulting homomorphism of groupschemes is (a model for) the abelianization of $GT_{\mathbb{Q}}$.

Now work of Hatcher and Bökstedt in differential topology [25][§6.4] constructs maps:

$$U/SO \longrightarrow B^2O \longleftarrow B(G/O) \longrightarrow Wh \longrightarrow \Omega^\infty S \times Wh = \Omega^\infty K(S)$$

of infinite loopspaces (where G is the monoid of homotopy self-equivalences of the stable sphere, and Wh is Waldhausen's Whitehead space of stable pseudoisotopies), which are \mathbb{Q} -isomorphisms on universal covers. The induced homomorphisms:

$$\begin{array}{ccc} K(S)_{*+1} \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_* B(U/SO) \otimes \mathbb{Q} = \pi_* Sp/U \otimes \mathbb{Q} \\ \downarrow \text{Borel} & & \downarrow \zeta \\ \mathbb{R} & \xrightarrow{\mathbb{Q}^\times} & \mathbb{R} \end{array}$$

on homotopy groups identify, up to rational units, Borel's regulators with the evaluation on symmetric functions defined above. This identification stabilizes to a rational equivalence:

$$k\zeta : S \vee \Sigma kO \cong_{\mathbb{Q}} K(S)$$

of spectra, defining an equivalence:

$$S \wedge_{S \vee \Sigma kO}^L S \cong_{\mathbb{Q}} S \wedge_{K(S)}^L S$$

and consequently an identification of $G \otimes \mathbb{Q}$ with the abelianization of

$$\text{Spec}(S \wedge_{K(S)}^L S) \otimes \mathbb{Q} \cong GT_{\mathbb{Q}}.$$

§5.3 Applications of homotopic descent

The constructions above fit very nicely with K. Hess's framework [24][§6] for descent theory in homotopy. To set this up, note that the Thom isomorphism defines a morphism

$$\begin{aligned} s\Omega(M) \wedge S[U/SO_+] &= s\Omega(M) \wedge_{s\Omega} s\Omega \wedge S[U/SO_+] \cong s\Omega(M) \wedge_{s\Omega} s\Omega \wedge (U/SO)^{-\zeta} \\ &= s\Omega(M) \wedge s\Omega \longrightarrow s\Omega(M) \end{aligned}$$

making $s\Omega(M)$ into a right module over the augmented S -algebra $S[U/SO_+]$. The point is that $s\Omega$ is Galois over S in the sense of [35], with $S[U/SO_+]$ as Hopf-Galois object, since

$$s\Omega \wedge s\Omega \cong s\Omega \wedge S[U/SO_+]$$

as ring-spectra. [This is a version of the normal basis theorem of classical Galois theory, which asserts that

$$F \otimes_E F \cong \text{Fns}(\text{Gal}(F/E), \mathbb{Q}) \otimes_{\mathbb{Q}} F$$

as algebras (for suitable extensions $F/E/\mathbb{Q}$).]

Since $S[U/SO_+]$ is augmented over S , the monoidal functor:

$$M \longmapsto s\Omega(M) \wedge_{S[U/SO_+]}^L S := s\Omega(M) \wedge_{S[U/SO_+]} Q(S) := \text{HH}_{\text{geo}}(M)$$

(with $Q(S)$ a cofibrant replacement for S as $S[U/SO_+]$ -module) is isomorphic, after tensoring with \mathbb{Q} , to $H_*(M, \mathbb{Q})$; we might thus think of it as a kind of geometric Hochschild homology of M . By Hess's work, this object has a canonical comodule structure:

$$s\Omega(M) \wedge_{S[U/SO_+]}^L S \longrightarrow (s\Omega(M) \wedge_{S[U/SO_+]}^L S) \wedge (S \wedge_{S[U/SO_+]}^L S)$$

over

$$S \wedge_{S[U/SO_+]}^L S \cong S[B(U/SO)_+].$$

It seems very likely that this algebraic structure on $\text{HH}_{\text{geo}}(M)$ agrees with the geometric structure defined by $s\Omega(M)$ as comodule over $\text{THH}(s\Omega)$; we intend to make this equivalence explicit in later work.

Finally, Kontsevich [29][§4.6.2, 8.4] shows that the graded cohomology algebra

$$H^*(X, \Lambda^* T)$$

of polyvector fields on a complex manifold is isomorphic to a kind of Hochschild cohomology

$$\text{HH}^*(X) = \text{Ext}_{\mathcal{O}_{X \times X}}^*(\mathcal{O}_X, \mathcal{O}_X)$$

defined in terms of the coherent sheaf of holomorphic functions on $X \times X$ and its diagonal. If X is Calabi-Yau, we can identify its tangent and cotangent bundle, obtaining an isomorphism of both cohomologies with the Hodge cohomology of X , defining [30] an action of (some version of) the Grothendieck-Teichmüller group on the de Rham cohomology of X . The homotopy-theoretic point of view suggests the Calabi-Yau hypothesis may be unnecessary.

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